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CERTAIN SELF-SIMILAR PROBLEMS OF FLOW OF A ONE-VELOCITY HETEROGENEOUS MEDIUM

V. S. Surov

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Solutions of self-similar problems of outflow of a mixture into vacuum, steady-state mixture flow near the exterior obtuse angle, and an accelerating piston in a dispersive medium have been obtained for the model of a one-velocity heterogeneous medium in which the internal forces of interfractional interaction are allowed for.

Two models of a one-velocity heterogeneous medium have been described in the literature. In the first model, the action of internal forces of interfractional interaction was disregarded [1], whereas in the second model, these forces were allowed for [2]. Their fundamental difference is that in the first model, both the entropy of the entire mixture and the entropies of the fractions composing the medium for an individual particle of the medium along its path remain constant, whereas in the second model, it is only the entropy of the entire mixture that is invariant. For the medium's model in which the interfractional-interaction forces were disregarded, we were able to obtain the analytical expression of the isentropy of the mixture and to write, on its basis, the Bernoulli integral (for stationary flows). These relations were subsequently used in constructing the solution of the problems on outflow of a dispersive medium into vacuum [3] and on steady-state flow of a multicomponent mixture near the exterior obtuse angle [4]. Clearly, the approaches developed in [3, 4] are inapplicable in the case where the model from [2] is used in solving the above-mentioned problems, since there is no analytical solution for the isentropy. Nonetheless, we are able to find the solutions of the problems indicated for this model, too, as will be shown below.

It is noteworthy that for the medium's model from [2], unlike [1], the Cauchy problem is correct for an arbitrary flow velocity and any number of fractions in the mixture (the corresponding dispersion equations have only real roots); therefore, the medium's model in which interfractional-interaction forces are allowed for is more preferable.

In describing the behavior of the components of the mixture, we use, for the sake of definiteness, the equation of state of the form

$$\varepsilon_i = \frac{p - c_{*i}^2 (\rho_i^0 - \rho_{*i})}{\rho_i^0 (\gamma_i - 1)} = \frac{b_i + pB_i}{\rho_i^0} - a_i \,, \tag{1}$$

where $B_i = 1/(\gamma_i - 1)$, $a_i = c_{*i}^2 B_i$, and $b_i = a_i \rho_{*i}$. When (1) is used, the equation of state of an n-component mixture with the first m compressible fractions takes the form [2]

$$\varepsilon = \frac{1}{\rho}\left[b_m + pB_m + \sum_{i=1}^{m-1} \alpha_i (b_{im} - a_{im}\rho_i^0 + pB_{im}) + \sum_{j=m+1}^{n} \alpha_j \rho_j^0 \varepsilon_j \right] - a_m \,. \tag{2}$$

Here $B_{im} = B_i - B_m$, $a_{im} = a_i - a_m$, and $b_{im} = b_i - b_m$.

Plane Steady-State Flow of the Medium near the Exterior Obtuse Angle. The system of governing equations which describes plane stationary flow of an n-component mixture with the first m compressible fractions is as follows [2]:

Chelyabinsk State University, 129 Brat'ya Kashiriny Str., Chelyabinsk, 454021, Russia; email: svs@csu.ru.
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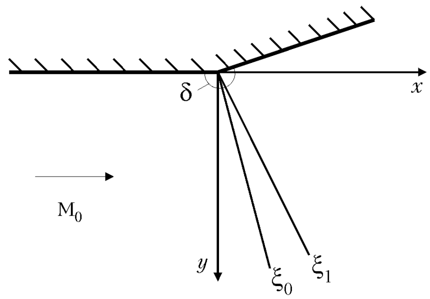

Fig. 1. Mixture flow near the exterior obtuse angle.

$$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}=0\,,\quad u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0\,,\quad u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=0\,,$$

$$A_1\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)+A_2\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)+\sum_{i=1}^{m-1}A_{i+2}\left(u\frac{\partial \rho_i^0}{\partial x}+v\frac{\partial \rho_i^0}{\partial y}\right)$$

$$+\sum_{i=1}^{m-1}A_{i+m+1}\left(u\frac{\partial \alpha_i}{\partial x}+v\frac{\partial \alpha_i}{\partial y}\right)+\sum_{j=m+1}^{n}A_{j+m}\left(u\frac{\partial \alpha_j}{\partial x}+v\frac{\partial \alpha_j}{\partial y}\right)=0\,,$$

$$-\frac{1}{\rho}\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)+\frac{1}{\rho_i^0}\left(u\frac{\partial \rho_i^0}{\partial x}+v\frac{\partial \rho_i^0}{\partial y}\right)+\frac{1}{\alpha_i}\left(u\frac{\partial \alpha_i}{\partial x}+v\frac{\partial \alpha_i}{\partial y}\right)=0\,, \tag{3}$$

$$\frac{B_i}{p}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)-\frac{b_i+p\,(1+B_i)}{p\rho_i^0}\left(u\frac{\partial \rho_i^0}{\partial x}+v\frac{\partial \rho_i^0}{\partial y}\right)-\frac{1}{\alpha_i}\left(u\frac{\partial \alpha_i}{\partial x}+v\frac{\partial \alpha_i}{\partial y}\right)=0\,,\quad i=1,\,...,\,m-1\,,$$

$$u\frac{\partial \alpha_j}{\partial x}+v\frac{\partial \alpha_j}{\partial y}+\frac{\alpha_j}{\rho}\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)=0\,,\quad j=m+1,\,...,\,n\,.$$

Here we have

$$A_1=-\frac{1}{\rho}\left[b_m+p\,(1+B_m)+\sum_{i=1}^{m-1}\alpha_i\,(b_{im}-\rho_i^0a_{im}+pB_{im})+\sum_{j=m+1}^{n}\alpha_jA_{j+m}\right]; \tag{4}$$

$$A_2=B_m+\sum_{i=1}^{m-1}\alpha_iB_{im}\,;\quad A_{i+2}=-\,\alpha_ia_{im}\,;\quad A_{i+m+1}=b_{im}-\rho_i^0a_{im}+pB_{im}\,;\quad A_{j+m}=\rho_j^0\varepsilon_j=\text{const}\,.$$

The solution of system (3) will be sought in the form $\rho=\rho(\xi)$, $u=u(\xi)$, $v=v(\xi)$, $p=p(\xi)$, $\rho_i^0=\rho_i^0(\xi)$, and $\alpha_i=\alpha_i(\xi)$, where $\xi=y/x$. In the physical plane $(x,\,y)$, the coordinate ξ corresponds to the straight line emergent from the origin of coordinates at an angle arctan (y/x) to the abscissa axis (Fig. 1), along which the values of the parameters of the mixture are constant. System (3) with account for the relations

$$\frac{\partial}{\partial x}=\frac{d}{d\xi}\frac{\partial \xi}{\partial x}=-\frac{\xi}{x}\frac{d}{d\xi}\,,\quad \frac{\partial}{\partial y}=\frac{d}{d\xi}\frac{\partial \xi}{\partial y}=\frac{1}{x}\frac{d}{d\xi}$$

is reduced to the system of ordinary differential equations

$$\rho \left(\frac{dv}{d\xi} - \xi \frac{du}{d\xi} \right) + (v - \xi u) \frac{d\rho}{d\xi} = 0 , \tag{5}$$

$$(v - \xi u) \frac{du}{d\xi} - \frac{\xi}{\rho} \frac{dp}{d\xi} = 0 , \tag{6}$$

$$(v - \xi u) \frac{dv}{d\xi} + \frac{1}{\rho} \frac{dp}{d\xi} = 0 , \tag{7}$$

$$A_1 \frac{d\rho}{d\xi} + A_2 \frac{dp}{d\xi} + \sum_{i=1}^{m-1} \left(A_{i+2} \frac{d\rho_i^0}{d\xi} + A_{i+m+1} \frac{d\alpha_i}{d\xi} \right) + \sum_{j=m+1}^{n} A_{j+m} \frac{d\alpha_j}{d\xi} = 0 , \tag{8}$$

$$- \frac{1}{\rho} \frac{d\rho}{d\xi} + \frac{1}{\rho_i^0} \frac{d\rho_i^0}{d\xi} + \frac{1}{\alpha_i} \frac{d\alpha_i}{d\xi} = 0 , \tag{9}$$

$$\frac{B_i}{p} \frac{dp}{d\xi} - \frac{b_i + p (1 + B_i)}{p\rho_i^0} \frac{d\rho_i^0}{d\xi} - \frac{1}{\alpha_i} \frac{d\alpha_i}{d\xi} = 0 , \quad i = 1, ..., m - 1 , \tag{10}$$

$$\frac{1}{\alpha_j} \frac{d\alpha_j}{d\xi} - \frac{1}{\rho} \frac{d\rho}{d\xi} = 0 , \quad j = m + 1, ..., n . \tag{11}$$

We seek the nontrivial solution of system (5)–(11). First we transform the original system. Multiplying Eq. (6) by $-\xi$ and adding to (7), we obtain the relation

$$\rho (v - \xi u) \left(\frac{dv}{d\xi} - \xi \frac{du}{d\xi} \right) + (1 + \xi^2) \frac{dp}{d\xi} = 0 ,$$

whose comparison to Eq. (5) yields

$$\frac{dp}{d\rho} = \frac{(v - \xi u)^2}{1 + \xi^2} . \tag{12}$$

Integrating Eqs. (9) and (11) from the initial state to the running one, we find

$$\alpha_i = \alpha_{i0} \frac{\rho}{\rho_0} \frac{\rho_{i0}^0}{\rho_i^0} , \quad i = 1, ..., m - 1 , \tag{13}$$

$$\alpha_j = \frac{\alpha_{j0} \rho}{\rho_0} , \quad j = m + 1, ..., n . \tag{14}$$

Taking relations (9)–(11) into account, we rewrite Eq. (8) in the form

$$\frac{dp}{d\rho} = \frac{A_1 + \dfrac{1}{\rho}\sum\limits_{i=1}^{m-1} \dfrac{\alpha_i\,[b_i + p\,(1 + B_i)]\,A_{i+m+1} - p\rho_i^0 A_{i+2}}{b_i + pB_i} + \dfrac{1}{\rho}\sum\limits_{j=m+1}^{n} \alpha_j A_{j+m}}{\sum\limits_{i=1}^{m-1} \dfrac{B_i\,(\alpha_i A_{i+m+1} - \rho_i^0 A_{i+2})}{b_i + pB_i} - A_2} . \tag{15}$$

A comparison of (15) to expression (12) yields another integral of system (5)–(11):

$$\frac{(v - \xi u)^2}{1 + \xi^2} = f\,(p, \rho, \rho_1^0, ..., \rho_{m-1}^0) , \tag{16}$$

where

$$f\,(p, \rho, \rho_1^0, ..., \rho_{m-1}^0) = \frac{\dfrac{1}{\rho}\left\{p\left[\sum\limits_{i=1}^{m-1} \dfrac{\alpha_i\,(b_{im} + pB_{im})}{b_i + pB_i} - (1 + B_m)\right] - b_m\right\}}{\sum\limits_{i=1}^{m-1} \dfrac{\alpha_i\,(b_{im}B_i - b_i B_{im})}{b_i + pB_i} - B_m} . \tag{17}$$

We must allow for expressions (13) in formula (17). Differentiating (16) with respect to ξ, we obtain the equation

$$\frac{dv}{d\xi} - \xi\,\frac{du}{d\xi} = \frac{u\,(v - \xi u) + \xi f}{v - \xi u} + D_1\,\frac{dp}{d\xi} + D_2\,\frac{d\rho}{d\xi} , \tag{18}$$

where

$$D_1 = \frac{1 + \xi^2}{2\,(v - \xi u)}\left(\frac{\partial f}{\partial p} + \sum_{i=1}^{m-1} \frac{\rho_i^0 B_i}{b_i + pB_i}\,\frac{\partial f}{\partial \rho_i^0}\right); \quad D_2 = \frac{1 + \xi^2}{2\,(v - \xi u)}\left(\frac{\partial f}{\partial \rho} - \frac{p}{\rho}\sum_{i=1}^{m-1} \frac{\rho_i^0}{b_i + pB_i}\,\frac{\partial f}{\partial \rho_i^0}\right).$$

Expressions for the derivatives $\partial f/\partial p$, $\partial f/\partial \rho$, and $\partial f/\partial \rho_i^0$ are not given because of their cumbersomeness. Equation (18) is considered in combination with (5)–(7) and with

$$\frac{d\rho_i^0}{d\xi} = \frac{\rho_i^0}{b_i + pB_i}\left(B_i\,\frac{dp}{d\xi} - \frac{p}{\rho}\,\frac{d\rho}{d\xi}\right), \quad i = 1, ..., m - 1 . \tag{19}$$

We reduce the new system of equations to a form convenient for integration:

$$\frac{dp}{d\xi} = -\frac{\rho\,(v - \xi u)\,[u\,(v - \xi u) + \xi f]}{\rho\,(v - \xi u)^2 D_1 + (1 + \xi^2)\,(v - \xi u + \rho D_2)} ,$$

$$\frac{du}{d\xi} = -\frac{\xi\,[u\,(v - \xi u) + \xi f]}{\rho\,(v - \xi u)^2 D_1 + (1 + \xi^2)\,(v - \xi u + \rho D_2)} ,$$

$$\frac{dv}{d\xi} = \frac{u\,(v - \xi u) + \xi f}{\rho\,(v - \xi u)^2 D_1 + (1 + \xi^2)\,(v - \xi u + \rho D_2)} ,$$

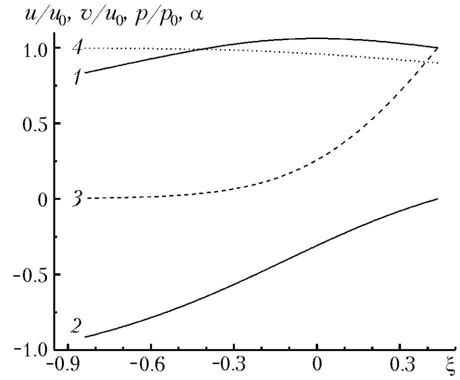

Fig. 2. Dependences of the parameters of flow of a binary mixture u/u_0 (1), v/v_0 (2), p/p_0 (3), and α (4) on ξ for M = 2.5.

$$\frac{d\rho}{d\xi} = -\frac{\rho\,(1+\xi^2)\,[u\,(v-\xi u)+\xi f]}{(v-\xi u)\,[\rho\,(v-\xi u)^2 D_1 + (1+\xi^2)\,(v-\xi u+\rho D_2)]}\,,$$

$$\frac{d\rho_i^0}{d\xi} = \frac{\rho_i^0\,[u\,(v-\xi u)+\xi f]\,[p\,(1+\xi^2)-\rho B_i\,(v-\xi u)^2]}{(b_i+pB_i)\,(v-\xi u)\,[\rho\,(v-\xi u)^2 D_1 + (1+\xi^2)\,(v-\xi u+\rho D_2)]}\,, \quad i=1,\ldots,m-1\,. \tag{20}$$

System (20) is integrated on the segment $[\xi_0, \xi_1]$, where $\xi_0 = 1/\sqrt{M_0^2 - 1}$ and $\xi_1 = -\tan\delta$; $M_0 = u_0/\sqrt{f(p_0, \rho_0, \rho_{10}^0, \ldots, \rho_{(m-1)0}^0)}$. The expression for ξ_0 follows from equality (17) if we set $p = p_0$, $u = u_0$, $v = 0$, $\rho = \rho_0$, and $\rho_1^0 = \rho_{10}^0$. The values of the volume concentrations of the fractions in the mixture are determined from relations (13) and (14) after the integration of system (20).

If some of the compressible fractions in the mixture are ideal gases for which $b_k = 0$, after the integration, Eqs. (19) for them will take the form

$$\rho_k^0 = \rho_{k0}^0 \frac{p}{p_0}\left(\frac{\rho_0}{\rho}\right)^{1/B_k}, \tag{21}$$

which diminishes the number of differential equations in system (20).

We note that if the angle δ is fairly large, there can occur a regime of flow to form a vacuum zone. The value of the coordinate ξ_* separating the region of flow of the dispersive medium from the vacuum zone, where $p = 0$, is determined from the condition $p(\xi_*) = 0$.

As an example of calculation from the relations given above, we calculate flow of a binary mixture of an ideal gas ($\gamma_1 = 1.4$ and $\rho_{10}^0 = 1.19$ kg/m^3) with an incompressible second component ($\rho_2^0 = 1000$ kg/m^3) near the angle $\delta = 220^{\circ}$. Figure 2 gives typical dependences of the distributions $u(\xi)/u_0$, $v(\xi)/u_0$, $p(\xi)/p_0$, and $\alpha(\xi)$, obtained in calculations of mixture flow with parameters $M_0 = 2.5$, $\alpha_0 = 0.9$, and $p_0 = 10^5$ Pa.

Outflow of the Multicomponent Medium into Vacuum. Governing equations for one-dimensional flow of an n-component mixture with the first m compressible fractions [2] have the form

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0\,,\quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0\,,$$

$$A_1\left(\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x}\right) + A_2\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) + \sum_{i=1}^{m-1}\left[A_{i+2}\left(\frac{\partial\rho_i^0}{\partial t} + u\frac{\partial\rho_i^0}{\partial x}\right) + A_{i+m+1}\left(\frac{\partial\alpha_i}{\partial t} + u\frac{\partial\alpha_i}{\partial x}\right)\right]$$

$$+\sum_{j=m+1}^{n} A_{j+m}\left(\frac{\partial \alpha_j}{\partial t}+u\,\frac{\partial \alpha_j}{\partial x}\right)=0\,,$$

$$-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}+u\,\frac{\partial \rho}{\partial x}\right)+\frac{1}{\rho_i^0}\left(\frac{\partial \rho_i^0}{\partial t}+u\,\frac{\partial \rho_i^0}{\partial x}\right)+\frac{1}{\alpha_i}\left(\frac{\partial \alpha_i}{\partial t}+u\,\frac{\partial \alpha_i}{\partial x}\right)=0\,, \tag{22}$$

$$\frac{B_i}{p}\left(\frac{\partial p}{\partial t}+u\,\frac{\partial p}{\partial x}\right)-\frac{b_i+p\,(1+B_i)}{p\rho_i^0}\left(\frac{\partial \rho_i^0}{\partial t}+u\,\frac{\partial \rho_i^0}{\partial x}\right)-\frac{1}{\alpha_i}\left(\frac{\partial \alpha_i}{\partial t}+u\,\frac{\partial \alpha_i}{\partial x}\right)=0\,, \quad i=1,\,...,\,m-1\,,$$

$$\frac{\partial \alpha_j}{\partial t}+u\,\frac{\partial \alpha_j}{\partial x}+\alpha_j\,\frac{\partial u}{\partial x}=0\,, \quad j=m+1,\,...,\,n\,.$$

The coefficients A_1, A_2, A_{i+2}, A_{i+m+1}, and A_{j+m} in (22) are calculated from formulas (4).

The solution of system (22) will be sought in the form $\rho = \rho(\xi)$, $u = u(\xi)$, $p = p(\xi)$, $\rho_i^0 = \rho_i^0(\xi)$, and $\alpha_i = \alpha_i(\xi)$, where $\xi = x/t$. By allowing for the relations

$$\frac{\partial}{\partial t}=\frac{d}{d\xi}\,\frac{\partial \xi}{\partial t}=-\frac{\xi}{t}\,\frac{d}{d\xi}\,, \quad \frac{\partial}{\partial x}=\frac{d}{d\xi}\,\frac{\partial \xi}{\partial \tilde{o}}=-\frac{1}{t}\,\frac{d}{d\xi}$$

it is reduced to a system of ordinary differential equations:

$$(u-\xi)\,\frac{d\rho}{d\xi}+\rho\,\frac{du}{d\xi}=0\,, \tag{23}$$

$$(u-\xi)\,\frac{du}{d\xi}+\frac{1}{\rho}\,\frac{dp}{d\xi}=0\,, \tag{24}$$

$$A_1\,\frac{d\rho}{d\xi}+A_2\,\frac{dp}{d\xi}+\sum_{i=1}^{m-1}\left(A_{i+2}\,\frac{d\rho_i^0}{d\xi}+A_{i+m+1}\,\frac{d\alpha_i}{d\xi}\right)+\sum_{j=m+1}^{n}\left(A_{j+m}\,\frac{d\alpha_j}{d\xi}\right)=0\,, \tag{25}$$

$$-\frac{1}{\rho}\,\frac{d\rho}{d\xi}+\frac{1}{\rho_i^0}\,\frac{d\rho_i^0}{d\xi}+\frac{1}{\alpha_i}\,\frac{d\alpha_i}{d\xi}=0\,, \tag{26}$$

$$\frac{B_i}{p}\,\frac{dp}{d\xi}-\frac{b_i+p\,(1+B_i)}{p\rho_i^0}\,\frac{d\rho_i^0}{d\xi}-\frac{1}{\alpha_i}\,\frac{d\alpha_i}{d\xi}=0\,, \tag{27}$$

$$-\frac{1}{\rho}\,\frac{d\rho}{d\xi}+\frac{1}{\alpha_j}\,\frac{d\alpha_j}{d\xi}=0\,. \tag{28}$$

We find the first integrals of system (23)–(28). After the integration of Eqs. (26) and (28) and transformations, we have

$$\alpha_i=\alpha_{i0}\,\frac{\rho_{i0}^0}{\rho_i^0}\,\frac{\rho}{\rho_0}\,, \tag{29}$$

$$\alpha_j = \frac{\alpha_{j0}\rho}{\rho_0} . \tag{30}$$

Eliminating $du/d\xi$ from relations (23) and (24), we obtain

$$\frac{dp}{d\rho} = (u - \xi)^2 . \tag{31}$$

Equation (25) with account for expressions (27) and (28) will take the form (15). From (31) and (15), we have another integral of system (23)–(28):

$$(u - \xi)^2 = f(p, \rho, \rho_1^0, ..., \rho_{m-1}^0) , \tag{32}$$

where the function $f(p, \rho, \rho_1^0, ..., \rho_{m-1}^0)$ is determined by formula (17), just as in the previous problem. Differentiating (32) with respect to ξ, we obtain the equation

$$\frac{du}{d\xi} = 1 + \frac{1}{2(u - \xi)} \left(D_1 \frac{dp}{d\xi} + D_2 \frac{d\rho}{d\xi} \right), \tag{33}$$

where

$$D_1 = \frac{\partial f}{\partial p} + \sum_{i=1}^{m-1} \frac{\rho_i^0 B_i}{b_i + pB_i} \frac{\partial f}{\partial \rho_i^0} ; \quad D_2 = \frac{\partial f}{\partial \rho} - \frac{p}{\rho} \sum_{i=1}^{m-1} \frac{\rho_i^0}{b_i + pB_i} \frac{\partial f}{\partial \rho_i^0} .$$

Equation (33) is considered in combination with Eqs. (23), (24), and (27). We reduce the new system of equations to a form convenient for integration:

$$\begin{gathered}
\frac{dp}{d\xi} = - \frac{2\rho (u - \xi)^3}{(u - \xi)^2 (2 + \rho D_1) + \rho D_2} , \\
\frac{du}{d\xi} = \frac{2 (u - \xi)^2}{(u - \xi)^2 (2 + \rho D_1) + \rho D_2} , \\
\frac{d\rho}{d\xi} = - \frac{2\rho (u - \xi)}{(u - \xi)^2 (2 + \rho D_1) + \rho D_2} , \\
\frac{d\rho_i^0}{d\xi} = - \frac{2\rho_i^0 (u - \xi) [\rho B_i (u - \xi)^2 - p]}{(b_i + pB_i) [(u - \xi)^2 (2 + \rho D_1) + \rho D_2]} , \quad i = 1, ..., m - 1 .
\end{gathered} \tag{34}$$

The system of ordinary differential equations (34) is integrated from the initial state with parameters $p = p_0$, $\rho = \rho_0$, $\rho_i^0 = \rho_{i0}^0$, and $u = u_0 = 0$ for $\xi = \xi_0$ to a certain ξ_* value that corresponds to the pressure $p = 0$. The value of ξ_0, in accordance with formula (32), is found from the relation

$$\xi_0 = - \sqrt{ \frac{1}{\rho_0} \left\{ b_m + p_0 \left[1 + B_m - \sum_{i=1}^{m-1} \frac{\alpha_{i0} (b_{im} + p_0 B_{im})}{b_i + p_0 B_i} \right] \right\} \Big/ \left[B_m + \sum_{i=1}^{m-1} \frac{\alpha_{i0} (b_i B_{im} - b_{im} B_i)}{b_i + p_0 B_i} \right] } = - c_0 ,$$

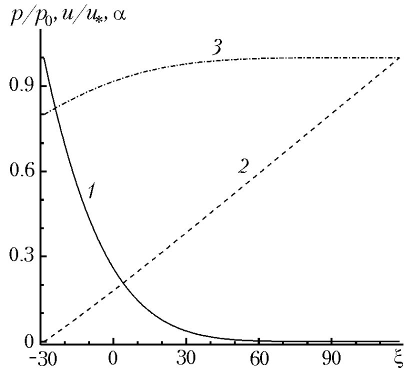

Fig. 3. Dependences of p/p_0 (1), u/u_* (2), and α (3) on ξ in outflow of a binary mixture into vacuum.

where c_0 is the velocity of sound in the mixture. The quantities α_i and α_j are determined from expressions (29) and (30) after the integration of system (34). If some of the compressible fractions in the mixture are ideal gases for which $b_k = 0$, relations (21) hold for them, which diminishes the number of differential equations in system (34). By solution of (34), we can also find the velocity of the "head" of the rarefaction-wave front, which is computed from the expression $u_* = u(\xi_*)$.

Figure 3 gives typical dependences of the distributions $p(\xi)/p_0$, $u(\xi)/u_*$, and $\alpha(\xi)$ in the gas-liquid mixture ($\gamma_1 = 1.4$, $c_{*1} = 0$, $\rho_{*1} = 1.19$ kg/m^3, $\gamma_2 = 5.59$, $c_{*2} = 1500$ m/sec, and $\rho_{*2} = 1000$ kg/m^3) for $p_0 = 10^5$ Pa, $\alpha_{10} = 0.8$, $\rho_{10}^0 = 1.19$ kg/m^3, and $\rho_{20}^0 = 1000$ kg/m^3. Integration of system (34) was from $\xi_0 = -28.65$ to $\xi_* = 105.88$; we had $u_* = 108.54$ m/sec.

Problem on the Accelerating Piston in a Dispersive Medium. We consider a fixed homogeneous gas-liquid mixture with a volume concentration of the gas α_0 and a density ρ_0 (the liquid fraction is assumed to be incompressible). At the initial instant of time $t = 0$, the piston begins to move at an accelerated pace under the action of external forces with a velocity whose law of variation with time has the form

$$u_{\rm p} = u_0 t^h .$$

The position of the piston in space is determined by the formula

$$x_{\rm p} = x_0 + \frac{u_0}{1+h} t^{1+h} ,$$

where x_0 is the initial coordinate of the piston. A shock wave is formed ahead of the piston.

It is expedient to analyze this problem in Lagrangian mass coordinates. It is necessary to find the solution of the system of equations

$$\frac{\partial \rho}{\partial t} - \rho^2 \frac{\partial u}{\partial \zeta} = 0 , \quad \frac{\partial u}{\partial t} + \frac{\partial p}{\partial \zeta} = 0 , \quad \frac{\partial p}{\partial t} + \frac{\gamma \rho p}{\alpha} \frac{\partial u}{\partial \zeta} = 0 , \quad \frac{\partial \alpha}{\partial t} - (1-\alpha) \rho \frac{\partial u}{\partial \zeta} = 0 \tag{35}$$

between the piston and the shock-wave front. Here the Lagrangian mass coordinate related to the Euler coordinate x by the relation $\zeta = \int_{x_0}^{x} \rho(y) dy$ is denoted by ζ. The solution of system (35) will be sought in the form

$$\rho = \rho_0 \rho(\xi) , \quad u = u_0 u(\xi) t^h , \quad p = \rho_0 u_0^2 p(\xi) t^{2h} , \quad \alpha = \alpha(\xi) , \quad \xi = \zeta / (\rho_0 u_0 t^{h+1}) . \tag{36}$$

Boundary conditions on the shock front are as follows [5]:

$$p_s = \frac{2\alpha_0\rho_0 D^2}{1+\gamma}, \quad \frac{\rho_s}{\rho_0} = \frac{1+\gamma}{1+\gamma-2\alpha_0}, \quad u_s = \frac{2\alpha_0 D}{1+\gamma}, \quad \alpha_s = \frac{\alpha_0(\gamma-1)}{1+\gamma-2\alpha_0}. \tag{37}$$

The shock wave is assumed to be "strong." Substituting expressions (36) into (35) and allowing for the relations

$$\frac{\partial}{\partial t} = \frac{d}{d\xi}\frac{\partial\xi}{\partial t} = -\frac{\xi(1+h)}{t}\frac{d}{d\xi}, \quad \frac{\partial}{\partial\zeta} = \frac{d}{d\xi}\frac{\partial\xi}{\partial\zeta} = \frac{1}{\rho_0 u_0 t^{1+h}}\frac{d}{d\xi}, \tag{38}$$

we obtain the system of four ordinary differential equations

$$-\xi(1+h)\frac{d\rho}{d\xi} + \rho^2\frac{du}{d\xi} = 0, \quad uh - \xi(1+h)\frac{du}{d\xi} + \frac{dp}{d\xi} = 0,$$

$$-\xi(1+h)\frac{dp}{d\xi} + \frac{\gamma\rho p}{\alpha}\frac{du}{d\xi} + 2ph = 0, \quad \xi(1+h)\frac{d\alpha}{d\xi} - (1-\alpha)\rho\frac{du}{d\xi} = 0. \tag{39}$$

Comparing the first and fourth relations of (39), we have

$$\frac{d\rho}{\rho} + \frac{d\alpha}{1-\alpha} = 0. \tag{40}$$

After the integration of (40), we obtain $\frac{\rho}{1-\alpha}$ = const; therefore, we have $\frac{\rho}{1-\alpha} = \frac{\rho_s}{1-\alpha_s}$. Whence, allowing for (38), we find a relationship between the volume concentration of the gas and the density of the medium:

$$\alpha = 1 - \rho(1-\alpha_0). \tag{41}$$

Using expression (41), we reduce system (39) to a form convenient for integration:

$$\frac{dp}{d\xi} = hp\frac{2\xi(1+h) + \frac{\gamma\rho u}{1-\rho(1-\alpha_0)}}{\xi^2(1+h)^2 - \frac{\gamma\rho p}{1-\rho(1-\alpha_0)}}, \quad \frac{du}{d\xi} = \frac{h}{\xi(1+h)}\left[u - p\frac{2\xi(1+h) + \frac{\gamma\rho u}{1-\rho(1-\alpha_0)}}{\xi^2(1+h)^2 - \frac{\gamma\rho p}{1-\rho(1-\alpha_0)}}\right],$$

$$\frac{d\rho}{d\xi} = \frac{h\rho}{\xi^2(1+h)^2}\left[u - p\frac{2\xi(1+h) + \frac{\gamma\rho u}{1-\rho(1-\alpha_0)}}{\xi^2(1+h)^2 - \frac{\gamma\rho p}{1-\rho(1-\alpha_0)}}\right]. \tag{42}$$

Allowing for the expression $D = d\zeta_s/dt = (1+h)\rho_0 u_0 \zeta_s t^h$, we rewrite the conditions on the shock-wave front (37):

$$\rho(\xi_s) = \frac{1+\gamma}{1+\gamma-2\alpha_0}, \quad p(\xi_s) = \frac{2\alpha_0(1+h)^2\xi_s^2}{1+\gamma}, \quad u(\xi_s) = \frac{2\alpha_0(1+h)\xi_s}{1+\gamma}. \tag{43}$$

Thus, on the segment $[0, \xi_s]$, we have the boundary-value problem for system (42) with the known values of $p(\xi_s)$, $u(\xi_s)$, and $\rho(\xi_s)$ on the shock front. Furthermore, at the point with a coordinate $\xi_p = 0$, where the piston is, its velocity $u_p = u_0 t_p^h$ is known. The last relation, with account for (36), will be rewritten as

$$u(0) = 1. \tag{44}$$

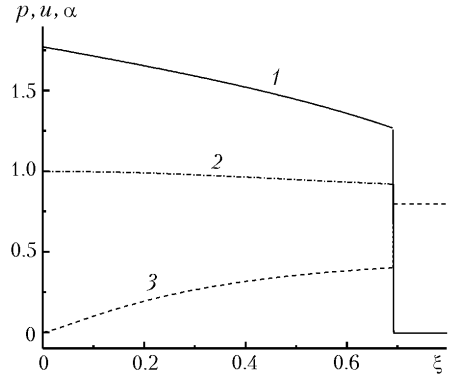

Fig. 4. Dependences of p (1), u (2), and α (3) on ξ for flow of a gas-liquid mixture.

We are unable to analytically solve problem (42)–(44). Numerical solution can be obtained in the following manner. We arbitrarily prescribe the position of the shock front ξ_s. We solve the Cauchy problem for system (42), starting from the shock front where the values of the functions are known from (43) to the point $\xi_p = 0$. We determine the velocity $u(0)$. If condition (44) is not observed, it is necessary to change ξ_s.

The typical distributions $p(\xi)$, $u(\xi)$, and $\alpha(\xi)$ obtained in calculations of flow of the gas-liquid mixture with parameters $h = 1$, $\alpha_0 = 0.8$, $u_0 = 10$ m/sec, $p_0 = 10^5$ Pa, $\gamma = 1.4$, $\rho_{10}^0 = 1.19$ kg/m^3, and $\rho_2^0 = 1000$ kg/m^3 are given in Fig. 4.

From the results of the work, we can draw the following conclusions. For the one-velocity heterogeneous-medium model, in which interfractional-interaction forces are allowed for, we have obtained the solutions of self-similar Prandtl–Meyer problems and problems of outflow of a mixture into vacuum and motion of a piston in a dispersive medium by the known law. In addition to being of independent importance, the solutions found in the work can also be used in testing computer programs intended for integration of the general equations of a one-velocity heterogeneous medium.

NOTATION

c, velocity of sound in the mixture, m/sec; c_{*i}, constant of the equation of state, m/sec; D, velocity of movement of the shock-wave front, m/sec; M, Mach number; m, number of compressible fractions in the mixture; n, total number of fractions in the mixture; p, pressure, Pa; u and v, projections of the velocity vector onto the x and y coordinate axes, m/sec; α_i, volume concentration of the ith component of the mixture; γ_i, constant of the equation of state; δ, opening span of the angle in flow, deg; ε, specific internal energy, m^2/sec^2; ζ, Lagrangian mass coordinate; ξ, self-similar variable; ρ, density of the mixture, kg/m^3; ρ_i^0, true density of the ith fraction, kg/m^3; ρ_i, reduced density of the ith component, kg/m^3; ρ_{*i}, constant of the equation of state, kg/m^3. Subscripts and superscripts: 0, in an unperturbed flow; s, on the shock front; p, on the piston.

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